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14. ABSTRACT

There have been a number of previous successful efforts that show how fuzzy logic concepts have homomorphic-like stochastic correspondences, utilizing one-point coverages of appropriately constructed random sets. Independent of this and fuzzy logic considerations in general, boolean relational event algebra (BREA) has been introduced within a stochastic setting for representing prescribed compositional functions of event probabilities by single compounded event probabilities. In the special case of the functions being restricted to division corresponding to pairs of nested sets, BREA reduced to boolean conditional event algebra (BCEA). BCEA has been successfully applied to issues involving comparing, contrasting and combining rules of inference, especially for those having differing antecedents. In this paper we show how, in a new way, not only BCEA, but also more generally, RCEA, can be applied to provide homomorphic-like connections between fuzzy logic quantifiers and classical logic relations applied to random sets. This also leads to an improved consistency criterion for these connections. Finally, when the above is specialized to BCEA, a novel extension of crisp boolean conditional events is obtained, compatible with the above improved consistency criterion.

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New applications of relational event algebra to fuzzy quantification and probabilistic reasoning

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Abstract

There have been a number of previous successful efforts that show how fuzzy logic concepts have homomorphic-like stochastic correspondences, utilizing one-point coverages of appropriately constructed random sets. Independent of this and fuzzy logic considerations in general, boolean relational event algebra (BREA) has been introduced within a stochastic setting for representing prescribed compositional functions of event probabilities by single compounded event probabilities. In the special case of the functions being restricted to division corresponding to pairs of nested sets, BREA reduced to boolean conditional event algebra (BCEA). BCEA has been successfully applied to issues involving comparing, contrasting and combining rules of inference, especially for those having differing antecedents. In this paper we show how, in a new way, not only BCEA, but also more generally, RCEA, can be applied to provide homomorphic-like connections between fuzzy logic quantifiers and classical logic relations applied to random sets. This also leads to an improved consistency criterion for these connections. Finally, when the above is specialized to BCEA, a novel extension of crisp

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boolean conditional events is obtained, compatible with the above improved consistency criterion.

1. Introduction

While certain foundation issues still remain, probability theory and its applications in conjunction with classical logic reasoning—even in a small sample size and/or subjective framework—have proven to be extremely useful tools in data fusion and related problems. A basic common direction of work that has evolved over the years among the authors of this paper is: a wide variety of real-world man–machine issues that have arisen in both military and non-military decision-making contexts can actually be fully successfully treated by standard probability theory and classical logic procedures, but utilizing new mathematical tools for expressing how standard probability theory can be so utilized. This is in contrast with many past and current views (either implicitly or explicitly expressed, e.g., in much of the artificial intelligence community, such as in [1]) that standard probability theory and classical logic are not adequate to treat these problems, and that, consequently, new formal logics and procedures incompatible with classical logic and standard probability theory are needed. In particular, the thesis of our past work has been to show how the tools of: (i) boolean conditional event algebra (BCEA) can be used to address appropriately in an ordinary probability setting problems of comparing and contrasting—as well as fusing and further analyzing in a rigorous, but implementable context—rules of inference [2]; (ii) more generally, boolean relational event algebra (BREA) can be used to compare or combine models of information input and output based upon "forcing functions" of probabilities of contributing events, such as weighted averages, negative exponentiation, or other simplifying numerical function of probabilities [2]; (iii) one point coverages of appropriately determined random sets (OPCRS) can be used to represent fuzzy logic modeling of information in an ordinary probabilistic setting, in order to further analyze, contrast, or combine such fuzzy logic models—especially those involving natural language information—with probabilistic-based information models. See, e.g., [2–5]; (iv) a second order bayesian probability approach (i.e., a bayesian approach where the relevant probabilities themselves are randomized) can be used to establish a fully rigorous and implementable probability logic that both extends classical entailment, is compatible in many ways with commonsense reasoning. In turn, this has led to the successfully solution of a number of long-open issues concerning the res-

olution of disparities between certain valid classical logic reasoning schemes and apparently straightforward probabilistic generalizations of these that seemingly become completely invalid. This includes the well-known schemes of *transitivity, contraposition*, and *strengthening of antecedent*, among many others. For an algorithmic form of this tool called Complexity-Reducing Algorithm for Near Optimal Fusion (CRANOF), see, e.g., [6]. It is expected that the first three tools BCEA, BREA, and OPCRS can be used to extend further the already wide applicability of CRANOF. To this end, [7] shows some preliminary applications of CRANOF to tracking and correlation of targets, including cyber-space intrusion and security issues.

In this very brief paper, we show two basic concepts connected with the above direction of work: (i) a simple motivating example for use of RCEA that can, in turn, be utilized in conjunction with CRANOF; (ii) an outline of how RCEA can be used in a new way to justify further in a rigorous sense the use of OPCRS in representing fuzzy logic concepts. More specifically, it is shown how a ubiquitous form of fuzzy logic quantification can be naturally interpreted in corresponding "pure" probabilistic terms.

Often, when only some of the relevant relations and events contributing to the understanding of a complex stochastic phenomenon are known, for purpose of simplicity, the output probability of the process may be modeled as a certain "forcing function" of the relevant input probabilities. Examples of such functions typically include weighted sums, products, divisions and conditionals, negative exponentiation, and various combinations of the above. When there is a multiplicity of choices for the forcing function determining the model at hand, the basic issue arises as to how to make the most appropriate choice, and hence, how to discriminate between choices, or equivalently, how to establish relevant measures/(pseudo) metrics measuring degrees of similarity or contrast among candidate forcing functions. At first thought, it would appear that a natural choice of metric is simply some distance function operating upon functions, such as the standard lebesgue p-norms or the sup-norm. However, this does not take into account the interaction/overlap of the structure of the models. To see this briefly, as described in more detail in [2], consider first comparing two given events a, b in a boolean algebra B. Letting $P : B \to [0, 1]$ be some relevant probability measure, one choice in comparing the events is simply using the absolute difference

$$d_{1,P}(a,b) =^d |P(a) - P(b)|. \tag{1.1}$$

But, as is almost obvious, one can have $d_{1,P}(a,b)$ quite small or even zero, yet a and b may be quite different in size, shape and location. At the very least, what is needed is that the metric here should account for overlap between the events themselves, not overlap of just their external probabilities, yet probability must also play a key role. A much more satisfactory metric (or pseudometric to be more precise) is $d_{2,P}$, where

$$d_{2,P}(a,b) =^d P(a \Delta b) = P(a'b) + P(ab') = P(a) + P(b) - 2 \cdot P(ab), \quad (1.2)$$

where we use, from now on, standard boolean algebra and probability notation, with Δ being the usual symmetric difference operator of B, &—or omission of any symbol, when no ambiguity arises—being the usual conjunction operator of B, $(\cdot)'$ being the usual complementation/negation operator of an event with respect to universal event Ω in B, $\vee$ being the usual disjunction operator of B, $\leqslant$ representing the usual subevent relation over B, not to be confused with the same symbol applied to numerical inequalities, etc. Note that, while the computation of $d_{2,P}(a,b)$ requires in part only knowledge of each $P(a)$, $P(b)$, it also utilizes, in a key way, knowledge of the interaction ab, and in turn, $P(ab)$. An even more satisfactory metric between events a and b is the conditional probability of symmetric difference, given disjunction, i.e., $d_{3,P}$, where

$$
\begin{aligned}
d_{3,P}(a,b) =^d\ & P(a \Delta b | a \vee b) \\
= & (P(a) + P(b) - 2 \cdot P(ab))/(P(a) + P(b) - P(ab)). \quad (1.3)
\end{aligned}
$$

For other candidate metrics and further discussion, again see [2]. Furthermore, if one wishes for any reason to determine interactions or disjunctive probabilities or any probability measure of logical relations between events a, b, depending on the logical structures of a and b, one can use all of the standard laws/properties of boolean algebra and probability theory to determine such computations. Finally, if one wishes, the choice of probability measure P can be randomized (thus, introducing *second order probability* concepts—see [6] for further discussion) and thus the above metrics become random variables with determined distributions, suitable here for testing of hypotheses, etc. This can be useful, when the choice of most appropriate P is not clear.

2. A basic example—weighted averages

Consider candidate Models 1, 2 with forcing functions in the form of some weighted average of probabilities:

$$
\begin{aligned}
&\text{Model 1}:\ P(C) = f_1(P(a), P(b)) =^d (1/2) \cdot P(a) + (1/2) \cdot P(b) \\
&\text{Model 2}:\ P(D) = f_2(P(a), P(b)) =^d (1/3) \cdot P(a) + (2/3) \cdot P(b).
\end{aligned} \quad (2.1)
$$

A (very simplified) example of this can arise when a represents the event (or proposition) "enemy will utilize area A", b represents "weather will be clear", c represents "enemy will take region B, according to Model 1", and d represents "enemy will take region B, according to Model 2", where the assigned weights for each model are determined by different experts in the field, integrating the two chosen contributing probabilities. It can be readily verified that natural

corresponding (but, by no means the only possible choices) *relational events* to Models 1, 2 are

$$f_1^*(a,b) =^d a \times b \times [0,1] \vee^* a \times b' \times [0,1/2] \vee^* a' \times b \times [0,1/2],$$
$$f_2^*(a,b) =^d a \times b \times [0,1] \vee^* a \times b' \times [0,1/3] \vee^* a' \times b \times [0,2/3], \qquad (2.2)$$

denoting ordinary cartesian product by $\times$, the real interval of numbers between, and including, r and s as $[r,s]$ (provided $r \leqslant s$), and disjunction in a new boolean algebra (see below) by $\vee^*$. This is because of the following: Let (Ω, B, P) be the given probability space, where a, b in boolean (or sigma) algebra B and P is used as in Eqs. (2.1) and (2.2). Then, construct the product probability space out of three factor probability spaces

$$(\Omega, B, P)^* =^d (\Omega^*, B^*, P^*)$$
$$=^d (\Omega, B, P) \otimes (\Omega, B, P) \otimes ([0,1], B[0,1], vol_1), \qquad (2.3)$$

where $B[0,1]$ is the real borel field of sets over the unit interval and vol_1 is ordinary one-dimensional lebesgue measure. Note the imbedding (i.e., isomorphic injection that is probability-preserving) $\psi : B \to B^*$, where, for all a in B,

$$\psi(a) =^d a \times \Omega \times [0,1]. \qquad (2.4)$$

Note also that using the mutual disjointness in Eq. (2.3),

$$P^*(f_1^*(a,b)) = P(a) \cdot P(b) + P(a) \cdot P(b') \cdot (1/2) + P(a') \cdot P(b) \cdot (1/2)$$
$$= (1/2) \cdot P(a) + (1/2) \cdot P(b), \qquad (2.5)$$

$$P^*(f_2^*(a,b)) = (1/3) \cdot P(a) + (2/3) \cdot P(b). \qquad (2.6)$$

Hence, we can identify for Models 1 and 2 in Eq. (2.1) C with $f_1^*(a,b)$ and D with $f_2^*(a,b)$, but where P on the left side of Eqs. (2.1) and (2.2) should really be replaced by P^*. In this vein, $\psi(a), \psi(b), f_1^*(a,b), f_2^*(a,b)$ all exist together in B^*. Note also that the evaluations in Eqs. (2.6) and (2.7) hold, regardless of the choice of specific P or particular pair a, b in B. Apropos to the above results, while the imbedding images of a, b exist together with the $f_j^*(a,b)$ in B^*, not only do a, b exist in B separate from $f_1^*(a,b), f_2^*(a,b)$ in B^*, but it is readily shown that, in general, there can be no events which play the same role as $f_1^*(a,b), f_2^*(a,b)$ also existing in B together with a, b. A special case of this is Lewis' triviality result [8] concerning conditional events (where the forcing function is arithmetic division operating upon nested pairs of events). For example, if for given a, b in B, with $a \neq b$, there is some c in B, with a, b, c fixed, so that for all choices of P Eq. (2.1) holds, easily is shown to lead to a contradiction. Hence, Eq. (2.1) holds for fixed c in B iff $c = a = b$.

Returning to the use of the $f_j^*(a,b)$ here in representing the respective models, one can now determine a meaningful probability distance between them, such as

$$
\begin{aligned}
d_{2,P^*}(f_1^*(a,b), f_2^*(a,b)) &= P^*(f_1^*(a,b)\Delta f_2^*(a,b)) \\
&= P^*(f_1^*(a,b)) + P^*(f_2^*(a,b)) \\
&\quad - 2\cdot P^*(f_1^*(a,b)\&^* f_2^*(a,b)) \\
&= f_1(P(a),P(b)) + f_2(P(a),P(b)) \\
&\quad - 2\cdot P^*(f_1^*(a,b)\&^* f_2^*(a,b)),
\end{aligned}
\tag{2.7}
$$

using Eqs. (2.6) and (2.7). Next, the conjunction $f_1^*(a,b)\&^* f_2^*(a,b)$ must be determined. Using the very definitions in Eq. (2.2), noting their disjoint term forms,

$$
\begin{aligned}
f_1^*(a,b)\&^* f_2^*(a,b) &= a\times b\times[0,1]\vee^* a\times b'\times([0,1/2]\cap[0,1/3]) \\
&\quad \vee^* a'\times b\times([0,1/2]\cap[0,2/3]) \\
&= a\times b\times[0,1]\vee^* a\times b'\times[0,1/3] \\
&\quad \vee^* a'\times b\times[0,1/2].
\end{aligned}
\tag{2.8}
$$

In turn, using the product probability property of P^*,

$$
\begin{aligned}
P^*(f_1^*(a,b)\&^* f_2^*(a,b)) &= P(a)\cdot P(b) + (1/3)\cdot P(a)\cdot P(b') \\
&\quad + (1/2)\cdot P(a')\cdot P(b).
\end{aligned}
\tag{2.9}
$$

Hence, substituting Eq. (2.9) into Eq. (2.7), finally yields

$$
\begin{aligned}
d_{2,P^*}(f_1^*(a,b), f_2^*(a,b)) &= (1/2)\cdot P(a) + (1/2)\cdot P(b) + (1/3)\cdot P(a) \\
&\quad + (2/3)\cdot P(b) - 2\cdot[P(a)\cdot P(b) \\
&\quad + (1/3)\cdot P(a)\cdot P(b') + (1/2)\cdot P(a')\cdot P(b)] \\
&= (1/6)\cdot P(a)\cdot P(b') + (1/6)\cdot P(a')\cdot P(b).
\end{aligned}
\tag{2.10}
$$

Note that if the naive distance function were used here, thus, not requiring any non-trivial application of BREA,

$$
\begin{aligned}
d_{1,P^*}(f_1^*(a,b), f_2^*(a,b)) &= |P^*(f_1^*(a,b)) - P^*(f_2^*(a,b))| \\
&= |f_1(P(a),P(b)) - f_2(P(a),P(b))| \\
&= (1/6)\cdot|P(a) - P(b)| \\
&= (1/6)\cdot|P(ab') - P(a'b)|.
\end{aligned}
\tag{2.11}
$$

Finally, note that essentially all of the relational event forms involving $a\times b$, $a'\times b$, $a\times b'$, could be replaced by the more interactive events (avoiding cartesian products) ab, $a'b$, ab' (but still retaining the cartesian products with the real interval events). In that interpretation, we have analogously

$$(\Omega, B, P)^* = (\Omega, B, P) \otimes ([0,1], B[0,1], vol_1),$$

$$f_1^*(a,b) =^d ab \times [0,1] \vee^* ab' \times [0,1/2] \vee^* a'b \times [0,1/2],$$

$$f_2^*(a,b) =^d ab \times [0,1] \vee^* ab' \times [0,1/3] \vee^* a'b \times [0,2/3],$$

$$d_{2,P^*}(f_1^*(a,b), f_2^*(a,b)) = (1/6) \cdot P(ab') + (1/6) \cdot P(a'b).$$

$$(2.12)$$

The above example leads to the general definition of a relational event and associated concepts:

Suppose n is a positive integer, there is a non-vacuous set $H \subseteq [0,1]^n$ and a function $f : H \to [0,1]$ such that for each given (non-trivial) probability space (Ω, B, P), there is another probability space $(\Omega, B, P)^* = (\Omega^*, B^*, P^*)$ (usually constructed out of appropriately determined cartesian products, as in the above example, etc.), and a non-vacuous set $K \subseteq B^n$ and mapping $f^* : K \to B^*$, such that the *relational event equations* hold:

(i) there is an imbedding $\psi : B \to B^*$ (isomorphic-probability preserving),

(ii) $f(P(a_1, \ldots, a_n)) = P^*(f^*(a_1, \ldots, a_n))$, for all $(a_1, \ldots, a_n)$ in K.

$$(2.13)$$

Thus, for the basic weighting average example above, $f_j^*(a,b)$ are special cases of $f^*(a_1, \ldots, a_n)$ in Eq. (2.13).

3. Homomorphic-like representations of fuzzy logic and fuzzy quantification

Theorem 1. *Let C be any given ordinary finite set ($\neq \emptyset$) and $\alpha_1, \ldots, \alpha_n : C \to [0,1]$ be any given fuzzy set membership functions. Let $\mathrm{cop} : [0,1]^C \times \cdots \times [0,1]^C \to [0,1]$ (n factors) be any copula. Thus, there is a probability space $(\Omega, B, P_{\mathrm{cop}})$ and random sets $S(\alpha_j, \mathrm{cop}) : \Omega \to \mathrm{Power}(C)$ such that the OPCRS holds* [5]

$$P_{\mathrm{cop}}(x \text{ in } \alpha_j) = \alpha_j(x), \quad \text{for all } x \text{ in } C, \ j = 1, \ldots, n. \tag{3.1}$$

Also, let $f : H \to [0,1]$ be any function representing some fuzzy logic quantification, where non-vacuous $H \subseteq [0,1]^n$, such as in the case of $n = 1$, for "very", "most", etc. Suppose also the relational event equations hold here for this choice of f, where all event n-tuples $((x_1 \text{ in } S(\alpha_1, \mathrm{cop})), \ldots, (x_n \text{ in } S(\alpha_n, \mathrm{cop})))$ are in K, for any choice of $\alpha_1, \ldots, \alpha_n$ above and any $x_1, \ldots, x_n$ in C. Then, the following homomorphic-like relations hold

$$P_{\mathrm{cop}}^*(f^*((x_1 \text{ in } S(\alpha_1, \mathrm{cop})), \ldots, (x_n \text{ in } S(\alpha_n, \mathrm{cop}))))$$

$$= f(P_{\mathrm{cop}}(x_1 \text{ in } S(\alpha_1, \mathrm{cop})), \ldots, P_{\mathrm{cop}}(x_n \text{ in } S(\alpha_n, \mathrm{cop})))$$

$$= f(\alpha_1(x_1), \ldots, \alpha_n(x_n)). \tag{3.2}$$

In particular, choosing the fuzzy sets α_j to be ordinary membership functions of crisp sets $a_j \subseteq C$, i.e., $\alpha_j = \phi(a_j)$, $j = 1,\ldots,n$, ϕ being the usual ordinary set membership functional, then regardless of choice of eop, $P_{\text{cop}} = P$, each $S(\phi(a_j),\text{cop}) = a_j$ and Eq. (3.2) specializes to

$$
\begin{aligned}
P^*(f^*(x_1 \text{ in } a_1),\ldots,(x_n \text{ in } a_n)) &= \phi(f^*(a_1,\ldots,a_n))(x_1,\ldots,x_n) \\
&= f(P(x_1 \text{ in } a_1),\ldots,P(x_n \text{ in } a_n)) \\
&= f(\phi(a_1)(x_1),\ldots,\phi(a_n)(x_n)).
\end{aligned}
\tag{3.3}
$$

As a basic application, consider briefly the BCEA form for any a, b in B, for $f(P(ab),P(b)) = P(ab)/P(b)$, where, $f(s,t) = s/t = s/[1-(1-t)] = \sum_{j=0}^{+\infty}(1-t)^j \cdot s$, taking $s = P(ab)$ and $t = P(b)$ suggests a natural algebraic counterpart, i.e., relational event, being the boolean conditional event

$$
f^*(a,b) =^d (a|b) =^d \bigvee_{j=0}^{+\infty}(\underbrace{b' \times \cdots \times b'}_{j \text{ factors}} \times a).
\tag{3.4}
$$

Indeed, it can be readily verified that when we choose the product probability space

$$
(\Omega,B,P)^* =^d (\Omega^*,B^*,P^*) =^d (\Omega,B,P)^{\otimes\infty} =^d (\Omega,B,P) \times (\Omega,B,P) \times \cdots
\tag{3.5}
$$

with the imbedding $\psi : B \to B^*$, where here $\psi(a) = (a|\Omega)$, for any a in B, the relational event equation holds, for any choice of probability measure $P : B \to [0,1]$,

$$
P^*((a|b)) = P(a|b), \quad \text{for } P(b) > 0.
\tag{3.6}
$$

For additional desirable algebraic and numerical properties of BCEA, see [9]. Choose any *radial symmetric* copula (see [10], where also the essentially equivalent concept of *survival* copula is provided) eop: $[0,1]^C \times [0,1]^C \to [0,1]$ and define copula eop_o: $([0,1]^C \times [0,1]^C) \times ([0,1]^C \times [0,1]^C) \times \cdots \to [0,1]$, where for any $x_1,x_2,x_3,\ldots$ in C,

$$
\text{cop}_o(x_1,x_2,x_3,\ldots) =^d \text{cop}(x_1,x_2) \cdot \text{cop}(x_3,x_4) \cdots
\tag{3.7}
$$

Choose $(\Omega,B,P)^* =^d (\Omega,B,P_{\text{cop}}) \times (\Omega,B,P_{\text{cop}}) \times \cdots$ Then, for any fuzzy set membership functions $\alpha,\beta : C \to [0,1]$, define fuzzy conditional event $(\alpha|\beta)_{\text{cop}} :$ $C \times C \times C \times \cdots \to [0,1]$ given as

$$(\alpha|\beta)_{\text{cop}_0}(x_1, x_2, x_3, \ldots)$$

$$=^d \sum_{j=0}^{+\infty} \left(\prod_{i=1}^{j} (\text{cop}(1 - \beta(x_i), 1)) \cdot \text{cop}(\alpha(x_{j+1}), \beta(x_{j+1})) \right)$$

$$= \sum_{j=0}^{+\infty} \prod_{i=1}^{j} (1 - \beta(x_i)) \cdot \text{cop}(\alpha(x_{j+1}), \beta(x_{j+1}))$$

$$= P^*_{\text{cop}}[\text{Or}(\text{And}(x_i \text{ in } (S(\beta, \text{cop})_i)'))$$

$$(i = 1, \ldots, j) \text{ and } (x_{j+1} \text{ in } S(\beta, \text{cop}) \cap S(\alpha, \text{cop}))],$$

indicating independent identical copies of the random sets

by subscript i,

$$= P^*_{\text{cop}}((x_1, x_2, x_3, \ldots) \text{ in } (S(\alpha, \text{cop})|S(\beta, \text{cop}))). \tag{3.8}$$

Next, a basic consistency relation holds when we specialize $\alpha = \phi(a)$, $\beta = \phi(b)$, for crisp sets $a, b \subseteq C$, Eq. (3.8) reduces to the form of the ordinary membership function of the conditional event expansion of $(a|b)$ as in Eq. (3.4). Finally, with appropriate modifications, all of the above can used as inputs to a wide variety of tracking and data fusion problems as in [7].

Acknowledgements

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